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# INVESTIGATIONS ON AN EDGE COLORING PROBLEM*

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**Abstract.** Given a bipartite graph $G$, and a sequence $H = (h_1, h_2, ..., h_n)$ of positive integers, necessary conditions have been given by Folkman and Fulkerson [1] for the existence of an edge coloring in which exactly $h_i$ edges have color $i$. These conditions are sufficient if $H$ has a particular form. In this paper, we determine necessary and sufficient conditions for the set $C$ of all color-feasible sequences (which is a partial order) to contain a unique maximal sequence. If $C$ has this property, the conditions of Folkman and Fulkerson are necessary and sufficient for a sequence $H$ to be color-feasible in $G$. Finally, an upper bound on the number of maximal sequences in $C$ is given when $G$ is bipartite and has maximum degree 3.

## § 1. Introduction

A graph $G = (X, U)$ consists of a finite nonempty set $X$ of vertices and a set $U$ of $m$ edges. An *edge coloring* $E(G;n)$ is a partition of $U$ into $n$ subsets $H_1, ..., H_n$ such that no two edges in the same $H_k$ are adjacent. Folkman and Fulkerson [2] have considered the following problem: When is a given finite sequence of positive integers $h_1, h_2, ..., h_n$ color-feasible in a given graph $G$? In other words, when does there exist an edge coloring $E(G;n)$ such that $H_i$ *has cardinality* $h_i$, $i = 1, 2, ..., n$? By using basic results of network flow theory they have obtained necessary and sufficient conditions for a sequence $h_1, h_2, ..., h_n$ containing no more than two distinct positive integers to be color-feasible in a bipartite graph.

These conditions are generally not sufficient for a sequence $H = h_1, h_2, ..., h_n$ to be color-feasible in a bipartite graph if $H$ has not the above mentioned property. If the set $C$ of all color-feasible se-

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quences in $G$ (which is a partially ordered set, as was observed by Folkman and Fulkerson [1]) contains a unique maximal sequence, then the conditions are sufficient. The purpose of this paper is to determine when $C$ has a unique maximal sequence. In the last section we will derive an upper bound on the number of maximal color-feasible sequences in a bipartite graph with maximum degree 3.

## § 2. Preliminary results

The *chromatic index* $n$ of a graph $G$ is the smallest number of subsets in an edge coloring of $G$. A *k-matching* $F$ in $G = (X, U)$ is a subset $F \subset U$ such that $G_F = (X, F)$ has chromatic index $n_F = k$. It is well known that the chromatic index of a bipartite graph $G$ is equal to the maximum degree of the vertices of $G$ [4]. A $k$-matching $F$ is *maximum* if there is no $k$-matching in $G$ with greater cardinality. If $G$ is a bipartite graph, a $k$-matching $F$ in $G$ corresponds simply to a subgraph $G_F = (X, F)$ with maximum degree $k$.

For $k = 1, ..., n$ let $f_k$ be the cardinality of a maximum $k$-matching in a bipartite graph $G$ with maximum degree $n$. Let the sequence $h_1^*, h_2^*, ..., h_n^*$ be defined by

(2.1) $\qquad h_1^* = f_1,\ h_i^* = f_i - f_{i-1},\ i = 2, 3, ..., n.$

**Lemma 2.1.** *The sequence $h_1^*, h_2^*, ..., h_n^*$ defined in* (2.1) *is such that* $h_1^* \geqslant h_2^* \geqslant ... \geqslant h_n^*$.

**Proof.** Let $A$ be the incidence matrix of the edges in $G$, i.e. $a_{ij} = 1$ if edge $j$ meets vertex $i$ and $a_{ij} = 0$ otherwise ($i = 1, ..., r; j = 1, ..., m$). Since $A$ is totally unimodular, $f_k$ is obtained by determining an optimal solution $x = (x_1, ..., x_m)$ of the linear programming problem $P(k)$

$$\max \{ z = \sum_{j=1}^{n} x_j : Ax \leqslant c,\ 0 \leqslant x_j \leqslant 1,\ j = 1, ..., m\} ,$$

where $c = (k, ..., k)$ has $r$ components.

Let $x^\ell = (x_1^\ell, ..., x_m^\ell)$ be an optimal solution of $P(\ell)$ for any $0 \leqslant \ell \leqslant n$. Clearly, $\frac{1}{2}x^{\ell-1} + \frac{1}{2}x^{\ell+1}$ is a feasible solution of $P(\ell)$; hence for $\ell = 1, ..., n-1$,

$$f_\ell = \sum_{i=1}^{m} x_i^\ell \geq \frac{1}{2} \sum_{i=1}^{m} x_i^{\ell-1} + \frac{1}{2} \sum_{i=1}^{m} x_i^{\ell+1} = \frac{1}{2} f_{\ell-1} + \frac{1}{2} f_{\ell+1},$$

or

$$h_\ell^* = f_\ell - f_{\ell-1} \geq f_{\ell+1} - f_\ell = h_{\ell+1}^*.$$

From now on, we will assume that in every sequence $h_1, h_2, \ldots, h_n$ the subscripts are chosen in such a way that $h_1 \geq h_2 \geq \ldots \geq h_n \geq 0$.

The following results appear in Folkman and Fulkerson [1]:

**Lemma 2.2.** *Let $G$ be an arbitrary graph; if $H = (h_1, h_2, \ldots, h_n)$ is color-feasible, then any sequence $H' = (h'_1, h'_2, \ldots, h'_n)$ such that*

$$\sum_{i=1}^{k} h'_i \leq \sum_{i=1}^{k} h_i, \quad k = 1, \ldots, n-1, \tag{2.2}$$

$$\sum_{i=1}^{n} h'_i = \sum_{i=1}^{n} h_i \tag{2.3}$$

*is color-feasible in $G$.*

If (2.2) and (2.3) hold for 2 sequences $H$ and $H'$, we will write $H > H'$. So the set $C$ of all color-feasible sequences in a graph $G$ is a partial order*. A sequence $H$ of $C$ is said to be *maximal* if there is no sequence $H' \neq H$ in $C$ with $H' > H$.

For a bipartite graph, we have the following result ([1]):

**Lemma 2.3.** *Where $G$ is a bipartite graph with maximum degree $n$, each maximal color-feasible sequence for $G$ contains exactly $n$ positive members.*

It is obvious that a necessary condition for a sequence $H = (h_1, h_2, \ldots, h_\ell)$ to be color-feasible in a bipartite graph $G$ with maxi-

* Formally all sequences in $C$ must have the same length $\ell$ ($\ell$ may be for instance the number of edges in $G$). However, we will omit the zero components in a sequence $H$ whenever no confusion is possible.

mum degree $n$ is that $H < H^* = (h_1^*, h_2^*, ..., h_n^*, 0, ..., 0)$ where the first $n$ components of $H^*$ are defined by (2.1) and the $\ell - n$ others are zero. These conditions are also sufficient if $H$ contains no more than two distinct positive integers as was shown by Folkman and Fulkerson [1]:

**Lemma 2.4.** *Let $H = (h_1, h_2, ..., h_\ell)$ be a sequence satisfying*

(2.4) $$h_1 = h_2 = ... = h_k = p,\ h_{k+1} = h_{k+2} = ... = h_\ell = q,$$

*then the sequence (2.4) is color-feasible in the bipartite graph $G$ if and only if*

(2.5) $$(h_1, h_2, ..., h_\ell) < (h_1^*, h_2^*, ..., h_n^*, 0, ..., 0),$$

*where the $h_i^*$'s are defined by (2.1).*

Condition (2.5) is generally not sufficient for $H$ to be color-feasible in $G$ if $H$ does not satisfy (2.4). It follows from Lemma 2.2 that these conditions are certainly sufficient if $H^* = (h_1^*, h_2^*, ..., h_n^*)$ is color-feasible in $G$. But this is equivalent to saying that the set $C$ of all color-feasible sequences in $G$ contains a unique maximal sequence. For suppose that $H = (h_1, ..., h_n) \neq H^*$ is the unique maximal sequence in $C$. So there exists an index $k$ such that $h_1 + ... + h_k < h_1^* + ... + h_k^* = f_k$. Let $F_k$ be a maximum $k$-matching in $G$; by partitioning $F_k$ into 1-matchings $H_1', ..., H_k'$ and $U - F_k$ into 1-matchings $H_{k+1}', ..., H_r'$ we obtain a color-feasible sequence $H' = (h_1', ..., h_r') \neq H$ for which $H' < H$ does not hold. This is a contradiction.

In the next section we will give necessary and sufficient condition for $C$ to have a unique maximal sequence.

## § 3. A theorem of uniqueness

First we have to introduce some more definitions. Given a bipartite graph $G$ with maximum degree $n$, let $H^* = (h_1^*, h_2^*, ..., h_n^*)$ be the unique sequence defined in (2.1). We associate with $H^*$ a sequence $S = (S(0), S(1), S(2), ..., S(t))$ defined as follows:

(a) $S(0) = 0$;

(b) for $j = 1, 2, 3, \ldots,$ $S(j)$ is the greatest index $\ell \leqslant n$ such that $h^*_{S(j-1)+1} - h^*_\ell \leqslant 1$.

For instance, if $H^* = (9,8,8,7,4,3)$ we will obtain $S = (0,3,4,6)$. $S$ will be called the *skeleton* of $H^*$ or simply of $G$. (We always have $S(t) = n$.)

**Lemma 3.1.** *Let $S = (S(0), S(1), S(2), \ldots, S(t))$ be the skeleton of a bipartite graph $G = (X, U)$; let $G' = (X, U')$ be a subgraph of $G$ such that for some $p$, $G'$ contains a maximum $S(p)$-matching of $G$ and a maximum $S(p+1)$-matching of $G$. Then for $k = S(p), S(p)+1, \ldots, S(p+1)$, $G'$ contains a maximum $k$-matching of $G$.*

**Proof.** From the definition of the skeleton we have

$$h^*_{S(p)+1} \geqslant h^*_{S(p)+2} \geqslant \cdots \geqslant h^*_{S(p+1)} \geqslant h^*_{S(p)+1} - 1.$$

Let $f'_k$ be the cardinality of a maximum $k$-matching in $G'$, and let $h'_k$ be defined by $h'_1 = f'_1$, $h'_i = f'_i - f'_{i-1}$, $i = 2,3,\ldots, S(p+1)$; let $a = S(p)$ and $b = S(p+1)$. From Lemma 2.1, we have

$$(3.1) \qquad h'_{a+1} \geqslant h'_{a+2} \geqslant \cdots \geqslant h'_b.$$

By hypothesis, $f_a = f'_a$ and $f_b = f'_b$. This implies

$$(3.2) \qquad h'_{a+1} + h'_{a+2} + \cdots + h'_b = h^*_{a+1} + h^*_{a+2} + \cdots + h^*_b.$$

Since $G'$ is a subgraph of $G$, we have $f'_k \leqslant f_k$, or

$$\sum_{i=1}^{k} h'_i \leqslant \sum_{i=1}^{k} h^*_i, \quad k = 1, \ldots, b,$$

and also

$$(3.3) \qquad h'_{a+1} + h'_{a+2} + \cdots + h'_{a+j} \leqslant h^*_{a+1} + h^*_{a+2} + \cdots + h^*_{a+j}.$$

We have now two cases to consider:

*Case 1.* $h^*_{a+1} = h^*_{a+2} = \ldots = h^*_b = h$. Suppose that for some $\ell$ ($a \leq \ell \leq b$) we have $f'_\ell < f_\ell$. Let $\ell$ be the first index for which $f'_\ell < f_\ell$. Then we would have $h'_\ell < h^*_\ell = h$. But (3.2) implies

$$h'_{\ell+1} + \ldots + h'_b > h^*_{\ell+1} + \ldots + h^*_b = h + \ldots + h.$$

Thus we have $h'_{\ell+1} > h$, but this is in contradiction with (3.1); hence $f'_k = f_k$ for $k = a, a+1, \ldots, b$.

*Case 2.* $h^*_{a+1} = h^*_{a+2} = \ldots = h^*_r = h$; $h^*_{r+1} = h^*_{r+2} = \ldots = h^*_b = h-1$. We show that we have

$$(3.4) \qquad h'_{a+1} + \ldots + h'_r = h^*_{a+1} + \ldots + h^*_r.$$

Suppose that

$$(3.5) \qquad h'_{a+1} + \ldots + h'_r < h^*_{a+1} + \ldots + h^*_r = h + \ldots + h.$$

This implies $h'_r < h$. But from (3.2) and (3.5) follows

$$h'_{r+1} + h'_{r+2} + \ldots + h'_b > h^*_{r+1} + \ldots + h^*_b = (h-1) + \ldots + (h-1).$$

Hence $h'_{r+1} > h - 1$. This contradicts (3.1); so (3.4) holds, and we are now in the situation of case 1. This ends the proof.

**Theorem 3.2.** *Let $S = (S(0), S(1), \ldots, S(t))$ be the skeleton of a bipartite graph $G = (X, U)$. The set $C$ of all color-feasible sequences in $G$ contains a unique maximal sequence if and only if there exists a family $F = \{F_1, F_2, \ldots, F_{t-1}\}$ of maximum $S(\ell)$-matchings $F_\ell$ in $G$ such that:*

$$(3.6) \qquad F_1 \subset F_2 \subset \ldots \subset F_{t-1}.$$

**Proof.** (A). If $C$ contains a unique maximal color-feasible sequence, then it is the sequence $H^* = (h^*_1, h^*_2, \ldots, h^*_n)$ defined in (2.1). So there exists a partition of $U$ into subsets $H^*_1, H^*_2, \ldots, H^*_n$ where $H^*_k$ has cardinality $h^*_k$ for $k = 1, \ldots, n$. For $\ell = 1, \ldots, t-1$ let $F_\ell$ be defined by

$$(3.7) \qquad F_\ell = \bigcup_{i=1}^{S(\ell)} H^*_i.$$

Each $F_\ell$ is a maximum $S(\ell)$-matching and the family $F$ thus obtained satisfies (3.6).

(B). We will prove by induction on the number $t + 1$ of components in the skeleton of $G$ that the existence of a family $F$ satisfying (3.6) implies that $C$ has a unique maximal color-feasible sequence. For $t + 1 = 2$, this is true: we have $S(t) = S(1) = n$ (where $n$ is the maximum degree in $G$) and so the sequence $H^* = (h_1^*, h_2^*, ..., h_n^*)$ satisfies (2.4) and (2.5). It follows from Lemma 2.4 that $H^*$ is color-feasible.

Thus $H^*$ is the unique maximal color-feasible sequence in $C$. Let us assume now that this result is true for bipartite graphs whose skeletons have at most $t$ components. Let $G = (X, U)$ be a bipartite graph, $S = (S(0), S(1), ..., S(t))$ its skeleton and $F$ a family of $t - 1$ maximal $S(\ell)$-matchings $F_\ell$ satisfying (3.6). Consider the graph $G' = (X, F_{t-1})$. As a consequence of Lemma 3.1 and of (3.6), the cardinality of a maximum $k$-matching in $G'$ is the same as in $G$ for $k = 1, ..., S(t - 1)$. Hence $S' = (S(0), S(1), ..., S(t - 1))$ is the skeleton of $G'$ and for the family $F' = \{F_1, F_2, ..., F_{t-1}\}$, (3.6) holds. So by our induction hypothesis, the set $C'$ of all color-feasible sequences in $G'$ has a unique maximal sequence $H' = (h_1^*, h_2^*, ..., h_{S(t-1)}^*)$.

We have to show now that there exists a family $F$ satisfying (3.6) and such that $U - F_{t-1}$ is a $(n - S(t - 1))$-matching. Assume that no such family exists; so for every $F$, $U - F_{t-1}$ is a $r$-matching with $r > n - S(t - 1)$. Hence every color-feasible sequence $H = (h_1, h_2, ..., h_p)$ with $h_i = h_i^*$, $i = 1, ..., S(t - 1)$ has more than $n$ positive members. On the other hand there is no color-feasible sequence $\overline{H}$ with exactly $n$ positive members which satisfies $\overline{H} > H$ (because this would imply the existence of a family $F$ such that $U - F_{t-1}$ is a $(n - S(t - 1))$-matching). Hence there exists in $C$ at least one maximal color-feasible sequence with more than $n$ positive members. This contradicts Lemma 2.3. So there is a family $F$ such that in $G'' = (X, U - F_{t-1})$ the maximum degree is $n - S(t - 1)$. Hence there exists a color-feasible sequence in $G''$ with $n - S(t - 1)$ positive members. It follows from Lemma 2.2 that $h_{S(t-1)+1}^*, ..., h_n^*$ which satisfies $h_{S(t-1)+1}^* - h_n^* \leq 1$ is color-feasible in $G''$ (it is the minimal color-feasible sequence in $G''$ among all sequences with exactly $n - S(t - 1)$ positive members). Thus $h_1^*, h_2^*, ..., h_n^*$ is color-feasible in $G$.

**Corollary 3.3.** *If the skeleton $S$ of a bipartite graph $G$ has at most 3 components, the set of all color-feasible sequences in $G$ contains a unique maximal sequence.*

**Proof.** If $S$ has 2 components, we are in the case of Lemma 2.4. If $S$ has 3 components, (3.6) is trivially satisfied ($F_1 \subset F_1$).

This result may also be expressed in the following way: Let $H = (h_1, h_2, ..., h_m)$ be a sequence satisfying

$$(3.8)\qquad \begin{array}{ll} h_1 = h_2 = ... = h_j = p, & h_{j+1} = h_{j+2} = ... = h_k = p-1, \\ h_{k+1} = h_{k+2} = ... = h_\ell = q, & h_{\ell+1} = h_{\ell+2} = ... = h_m = q-1, \end{array}$$

then condition (2.5) is necessary and sufficient for $h_1, h_2, ..., h_m$ to be color-feasible in a bipartite graph $G$.

We conclude this section with some remarks on Theorem 3.2.

*Remark 1.* It is interesting to observe that in the family $F$ each $F_\ell$ is simply a maximal $S(\ell)$-matching; it may not be an *admissible* $S(\ell)$-matching, i.e. an $S(\ell)$-matching such that $U - F_\ell$ is an $(n - S(\ell))$-matching and hence may be colored with $(n - S(\ell))$ colors. (Notice that the $F_\ell$ defined in (3.7) is of course admissible.)

*Remark 2.* If $G$ is not bipartite, Lemma 2.3 may not be true; as a consequence it may happen that there is no maximum $k$-matching which is admissible. (If $G$ has chromatic index $n$, a $k$-matching $F$ is admissible if $G' = (X, U - F)$ has chromatic index $n - k$.) Consider for instance the graph $G_1$ in fig. 1 for which the unique maximum 1-matching $F_1$ has

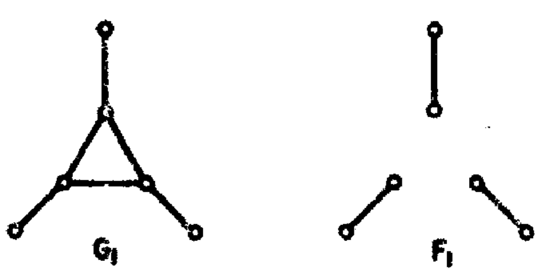


Fig. 1.

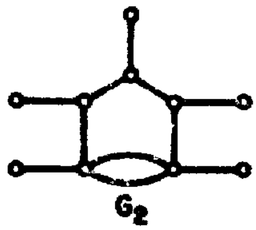


Fig. 2.

cardinality 3, while every admissible 1-matching has cardinality 2. (The maximal sequences in $C$ are (3, 1, 1, 1) and (2, 2, 2).)

Suppose we define $f_k$ as the largest possible cardinality for an admissible $k$-matching and $h_k^*$ as in (2.1). The case where $h_k^* < h_{k+1}^*$ for some $k$ may occur. As an example, consider graph $G_2$ in fig. 2; we have $(h_1^*, h_2^*, h_3^*, h_4^*) = (5, 2, 3, 1)$. The maximal color-feasible sequences in $G$ are (5, 2, 2, 2) and (4, 3, 3, 1). This means that it is not always possible to define a skeleton $S$ as for bipartite graphs. However, even if $h_1^* \geqslant h_2^* \geqslant \ldots \geqslant h_n^*$ holds and if there exists a maximum $k$-matching which is admissible for $k = 1, \ldots, n$, Theorem 3.2 may not be true for non bipartite graphs. Examine the graph $G_3$ in fig. 3. $F_1 = H_1$ is a maximum and admissible 1-matching; $H_1' \cup H_2'$ is a maximum and admissible 2-matching, while $F_2 = H_1' \cup H_2' \cup H_3'$ is a maximum and admissible

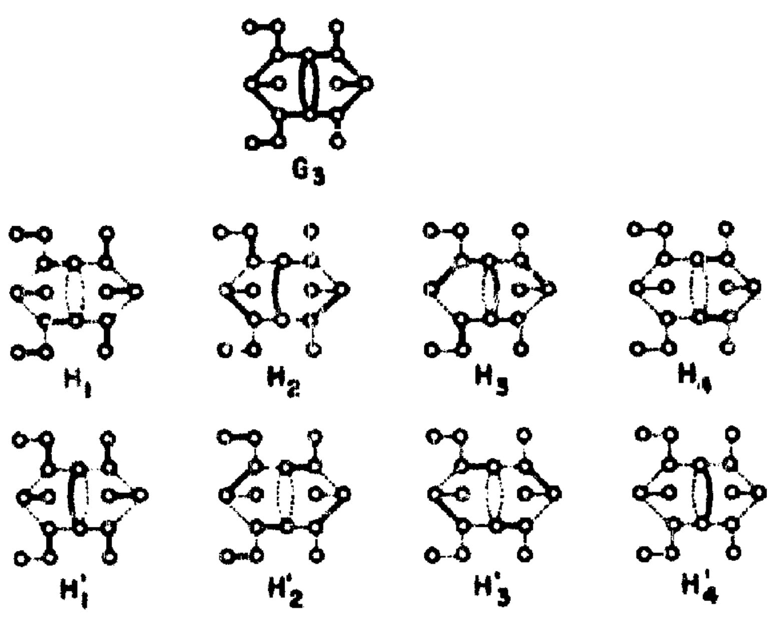


Fig. 3.

3-matching. We have $(h_1^*, h_2^*, h_3^*, h_4^*) = (8, 5, 4, 1)$ and the skeleton $S$ is $(0, 1, 3, 4)$; condition (3.6) becomes $F_2 \supset F_1$ and one verifies that it is satisfied. However, $(8, 5, 4, 1)$ is not color-feasible in $G_3$. The maximal sequences are $(8, 4, 4, 2)$ and $(7, 6, 4, 1)$ and they correspond to the edge colorings represented in fig. 3.

## § 4. On the number of maximal color-feasible sequences in $C$

In the previous section we have obtained conditions for the set $C$ of all color-feasible sequences in a bipartite graph $G$ to contain a unique maximal sequence.

Our attempts to determine an upper bound for the number of maximal color-feasible sequences in a bipartite graph with maximum degree $n$ and with $m$ edges have not been successful. However this has been done in the case $n = 3$. (Note that if $n = 2$, $C$ always has a unique maximal sequence from Lemma 2.4).

**Lemma 4.1.** *In a connected graph $G$ with $m$ edges, the cardinality $h_1^*$ of a maximum 1-matching satisfies $h_1^* \leq \frac{1}{2}(m+1)$.*

**Proof.** Given a maximum 1-matching $H$ in $G$ with cardinality $h_1^*$, $G$ may be connected only if there are at least $h_1^* - 1$ edges of $G$ which are not in $H$. Hence $m \geq 2h_1^* - 1$.

**Lemma 4.2.** *Let $G$ be a bipartite graph with maximum degree 3; let $H^* = (h_1^*, h_2^*, h_3^*)$ be the sequence defined in (2.1). If $\min(h_1^* - h_2^*, h_2^* - h_3^*) = \ell$, then $C$ contains at most $[\frac{1}{2}\ell] + 1$ maximal color-feasible sequences.*

**Proof.** (1). Suppose $h_1^* - h_2^* = \ell$. Then the sequence $\bar{H} = (\bar{h}_1, \bar{h}_2, \bar{h}_3)$ where

$$\bar{h}_1 = h_1^* - [\tfrac{1}{2}\ell], \; \bar{h}_2 = h_2^* + [\tfrac{1}{2}\ell], \; \bar{h}_3 = h_3^* \tag{4.1}$$

satisfies (2.5) and (3.8). Hence $\bar{H}$ is color-feasible in $G$. Furthermore, $\bar{h}_1 \geq \bar{h}_2 \geq \bar{h}_1 - 1$.

(2). The number $M$ of maximal sequences in $C$ is not greater than the maximum number of mutually unrelated sequences in $C$. ($H'$ and $H''$ are unrelated in $C$ if neither $H' > H''$ nor $H'' > H'$ holds.)

(3) Let $C(\bar{H})$ be the set of all color-feasible sequences $H$ in $C$ for which $H < \bar{H}$. Since $\bar{H}$ is color-feasible, no sequence in $C(\bar{H})$ is a maximal sequence. Hence $M$ is not greater than the maximum number of unrelated sequences in $C - C(\bar{H})$.

(4). Every sequence $H = (h_1, h_2, h_3)$ in $C - C(\bar{H})$ satisfies $h_1 \geq h_1^* - [\frac{1}{2}\ell]$: If $h_1 < h_1^* - [\frac{1}{2}\ell]$, then we must have $h_2 \leq h_2^* + [\frac{1}{2}\ell]$ (otherwise from (1) we would have $h_1 < h_2$ which is not possible). But such a sequence $H$ would satisfy $H < \bar{H}$ and there is no such sequence in $C - C(\bar{H})$.

(5). For every sequence $H$ in $C - C(\bar{H})$, $h_1$ can take at most $[\frac{1}{2}\ell]+1$ different values. Furthermore, if for two sequences $H = (h_1, h_2, h_3)$ and $H' = (h_1', h_2', h_3')$ we have $h_1 = h_1'$, $H$ and $H'$ are not unrelated. Hence there are at most $[\frac{1}{2}\ell]+1$ unrelated sequences in $C - C(\bar{H})$. The case $h_2^* - h_3^* = \ell$ can be dealt with similarly.

**Theorem 4.3.** *Given any sequence $H^* = (h_1^*, h_2^*, h_3^*)$ of positive integers $(h_1^* \geq h_2^* \geq h_3^*)$, satisfying $h_2^* + h_3^* \geq h_1^* - 1$; let $\ell = \min(h_1^* - h_2^*, h_2^* - h_3^*)$. Then there exists a bipartite connected graph G for which*

(a) *$H^*$ is the sequence defined in (2.1),*

(b) *the set of all color-feasible sequences in G contains $[\frac{1}{2}\ell]+1$ maximal sequences.*

**Proof.** Let us define $p = h_2^* + h_3^* - h_1^* + 1$, $q = h_1^* - h_2^* - 2[\frac{1}{2}\ell]$, $r = h_2^* - h_3^* - 2[\frac{1}{2}\ell]$. By assumption, $p \geq 0$ (if $p < 0$, it follows from Lemma 4.1 that the sequence $H^*$ cannot correspond to a connected graph); and by the definition of $\ell$ we also have $q, r \geq 0$. The graph $G$ is obtained by combining $[\frac{1}{2}\ell]$ graphs $I_1, I_2, \ldots, I_{[\frac{1}{2}\ell]}$, $[\frac{1}{2}\ell] - 1$ graphs $J_1, J_2, \ldots, J_{[\frac{1}{2}\ell]-1}$, $p$ graphs $K_1, K_2, \ldots, K_p$, $q$ graphs $L_1, L_2, \ldots, L_q$ and $r$ graphs $M_1, M_2, \ldots, M_r$, as shown in fig. 4.

One verifies easily that the maximal cardinality of a 1-matching $F_1$ is $h_1^*$: $F_1$ contains 5 edges of each $I_i$, 1 edge of each $J_i$, 1 edge of each $K_i$, 2 edges of each $L_i$, and 1 edge of each $M_i$. So $F_1$ has cardinality $5[\frac{1}{2}\ell] + ([\frac{1}{2}\ell] - 1) + p + 2q + r = h_1^*$. Similarly one verifies that the maximal cardinality of a 2-matching is $h_1^* + h_2^*$ and that $G$ has $m = h_1^* + h_2^* + h_3^*$ edges.

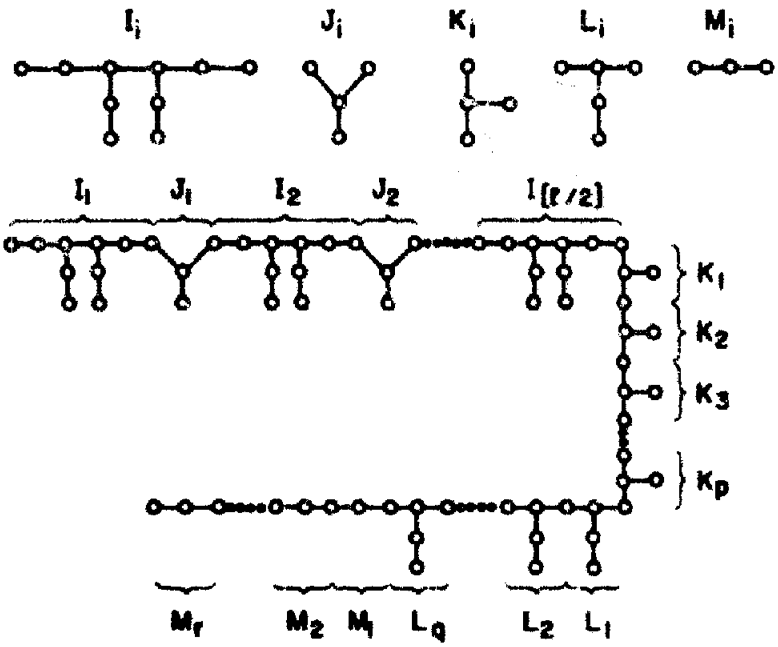


Fig. 4.

Furthermore, let $F_1$ be any maximal (and admissible) 1-matching in $G$ and let $F_2$ be a 2-matching containing $F_1$ and having cardinality as large as possible. $F_2$ has cardinality $h_1^* + h_2^* - [\frac{1}{2}\ell]$. There are $[\frac{1}{2}\ell]$ disjoint alternating chains $P_i$ (one in each $I_i$) which may be used to increase the cardinality of $F_2$. By using any $P_i$, the cardinality of $F_1$ is decreased by one unit. Hence, $C$ has $[\frac{1}{2}\ell]+1$ maximal sequences

$$
\begin{array}{lll}
(h_1^*, & h_2^* - [\frac{1}{2}\ell], & h_3^* + [\frac{1}{2}\ell]),\\
(h_1^* - 1, & h_2^* - [\frac{1}{2}\ell] + 2, & h_3^* + [\frac{1}{2}\ell] - 1),\\
(h_1^* - i, & h_2^* - [\frac{1}{2}\ell] + 2i, & h_3^* + [\frac{1}{2}\ell] - i),\\
(h_1^* - [\frac{1}{2}\ell], & h_2^* + [\frac{1}{2}\ell], & h_3^*).
\end{array}
$$

**Theorem 4.4.** *Let $m$ be the number of edges in a connected bipartite graph with maximum degree 3. The set $C$ of color-feasible sequences in $G$ contains at most $[\frac{1}{12}(m+15)]$ maximal sequences.*

**Proof.** Let $H^* = (h_1^*, h_2^*, h_3^*)$ be the sequence (2.1). The upper bound of Lemma 4.2 on the number of maximal sequences is maximum when

$\ell$ is maximum, i.e. $|(h_1^* - h_2^*) - (h_2^* - h_3^*)| \leqslant 1$. This implies $h_2^* = [\frac{1}{3}m]$. From Lemma 4.1 we have $h_1^* \leqslant \frac{1}{2}(m+1)$. So the maximum number of maximal sequences in $C$ is

$$\tfrac{1}{2}[h_1^* - h_2^*] + 1 \leqslant \tfrac{1}{2}([\tfrac{1}{2}(m+1)] - [\tfrac{1}{3}m]) + 1 \leqslant [\tfrac{1}{12}(m+15)]\,.$$

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